MATH 120A Prep: Modular Arithmetic

1. Write the elements for $\mathbb{Z}_5$ then create an addition and multiplication table for $\mathbb{Z}_5$.

Solution: The elements of $\mathbb{Z}_5$ are [0], [1], [2], [3], and [4]. The addition and multiplication tables are:

+	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

x	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]
[2]	[0]	[2]	[4]	[1]	[3]
[3]	[0]	[3]	[1]	[4]	[2]
[4]	[0]	[4]	[3]	[2]	[1]

2. Let $[a]_n$ denote the equivalence class of a in the set $\mathbb{Z}_n$. Define a function $f : \mathbb{Z}_9 \to \mathbb{Z}_3$ by $f([a]_9) = [a]_3$. Show this map is well-defined and write out where each element of $\mathbb{Z}_9$ maps to in $\mathbb{Z}_3$. What are the elements of $\mathbb{Z}_9$ that map to $[0]_3$?

Solution: Well-defined: Suppose that $[a]_9 = [b]_9$, so $9|(a-b)$. Therefore $3|(a,b)$ and so $[a]_3 = [b]_3$ which is what we want since then $f([a]_9) = f([b]_9)$.

This maps each elements of $\mathbb{Z}_9$ by:

$$f([0]_9) = [0]_3$$
$$f([1]_9) = [1]_3$$
$$f([2]_9) = [2]_3$$
$$f([3]_9) = [0]_3$$
$$f([4]_9) = [1]_3$$
$$f([5]_9) = [2]_3$$
$$f([6]_9) = [0]_3$$
$$f([7]_9) = [1]_3$$
$$f([8]_9) = [2]_3$$

Then the elements that map to $[0]_3$ are $[0]_9$, $[3]_9$, and $[6]_9$. It's worth noting that these correspond to multiples of 3.

3. Using the same function from Problem 2, prove that $f([a]_9 + [b]_9) = f([a]_9) + f([b]_9)$.

Solution:

$$f([a]_9 + [b]_9) = f([a+b]_9) = [a+b]_3 = [a]_3 + [b]_3 = f([a]_9) + f([b]_9)$$

While this problem may appear straight-forward this is actually an important property to have in algebra.